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Diagonalization of operators and one-parameter families of nonstandard bases for representations of $su_q(2)$

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Abstract
Nonstandard bases for finite dimensional irreducible representations of the quantum algebra $su_q(2)$ are constructed by diagonalizing one-parameter families of the operators $q^{J_3/4}(J_+ + J_-)q^{J_3/4} + cq^{J_3}$ and $\mathrm{i}q^{J_3/4}(J_+ - J_-)q^{J_3/4} + cq^{J_3}$, $c \in \mathbb{R}$. We derive explicit expressions for the eigenfunctions and the corresponding eigenvalues of these operators in an arbitrary irreducible representation of $su_q(2)$. It is shown that the matrix elements of the intertwining operator $A^j(c)$, which is a q-extension of the classical $su(2)$-operator a^j, $J_1a^j = a^jJ_3$, are expressed in terms of the dual q-Krawtchouk polynomials. Diagonalization of some other operators, associated with the dual q-Hahn polynomials, is also examined.

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1. Introduction

In [1] an explicit expression for the eigenfunctions and the corresponding eigenvalues of the operator $\widetilde{J}_1 \equiv \left(q^{1/4}J_+ + q^{-1/4}J_-\right)q^{J_3/2}$ in an arbitrary finite dimensional irreducible representation of the quantum algebra $su_q(2)$ was derived. The basis consisting of these eigenfunctions is called *nonstandard* in order to distinguish it from the standard *canonical* basis, which is formed by the eigenfunctions of the operator J_3. It was shown in [1] that the canonical and nonstandard bases are connected by a matrix with entries, expressed in terms of the dual q-Krawtchouk polynomials of a special type. A motivation for studying operators such as $\widetilde{J}_1$ comes from mathematical and theoretical physics. Many models in quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians of the form $\widetilde{J}_1$ (see, e.g., [2] and references therein). A complementary motivation for studying different bases arises from mathematics, namely, from the theory of special functions and in particular orthogonal polynomials [3–6] (see also [7, 8]). Studies in q-analysis and q-special functions are important in the construction of a renovated background for field theory model building.

One can diagonalize more general representation operators by using the same technique as in [1]. We show in the present paper that the one-parameter family of operators $\check{J}_1^{(c)} = \widetilde{J}_1 + cq^{J_3}$, $c \in \mathbb{R}$, in an arbitrary irreducible representation of $su_q(2)$ can also be diagonalized. We find an explicit form of eigenfunctions of these operators and the corresponding eigenvalues. For each value of c, the eigenfunctions of the operator $\check{J}_1^{(c)}$ constitute a *nonstandard* basis of the representation space. We thus derive a one-parameter family of nonstandard bases for each finite dimensional irreducible representation of the quantum algebra $su_q(2)$. We show that these nonstandard bases are connected with the canonical basis by a matrix with entries expressed in terms of more general dual q-Krawtchouk polynomials than in the case of the operator $\widetilde{J}_1$.

Further, we derive, for each value of c, an explicit formula for the action of the operator q^{-J_3} upon the corresponding nonstandard basis. This action is given by a Jacobi matrix (that is, a matrix with only three nonvanishing diagonals).

Having an explicit form of the operators $\check{J}_1^{(c)}$ and q^{-J_3}, one may find (by multiplying them and using their linear combinations) an explicit form of many other representation operators. However, we could not find an explicit form for the action of the operators q^{J_3}, $q^{J_3/2}$, $q^{-J_3/2}$. If we could obtain the last operators, we would be able to calculate any representation operator.

We also show that some other representation operators of $su_q(2)$ can be diagonalized by means of the dual q-Hahn polynomials. These operators also form a one-parameter family of representation operators. We have explicitly constructed the corresponding one-parameter family of nonstandard bases.

The results of the paper are true for any complex value of q not equal to a root of unity. We only assume that q is positive when we use the orthogonality relation for the dual q-Krawtchouk polynomials.

2. The algebra $su_q(2)$ and its representations

For any fixed complex value of q, the algebra $su_q(2)$ can be defined as an associative algebra generated by the elements J_1, J_2 and J_3, satisfying the relations

$$[J_1, J_2] = \frac{\mathrm{i}}{2}[2J_3]_q \qquad [J_2, J_3] = \mathrm{i}J_1 \qquad [J_3, J_1] = \mathrm{i}J_2 \tag{2.1}$$

where

$$[A]_q := \frac{q^{A/2} - q^{-A/2}}{q^{1/2} - q^{-2/2}}.$$

In terms of the raising $J_+ = J_1 + \mathrm{i}J_2$ and lowering $J_- = J_1 - \mathrm{i}J_2$ operators relations (2.1) take the form

$$[J_+, J_-] = [2J_3]_q \qquad [J_3, J_\pm] = \pm J_\pm.$$

Nontrivial finite dimensional irreducible representations of the algebra $su_q(2)$ are given by positive integers or half-integers j. We denote these representations by T_j.

The linear space of the irreducible representation T_j can be realized as the space $\mathcal{H}_j$ of all polynomials in x of degree less or equal to $2j$. The operators J_3 and $J_\pm$ are realized in this space as

$$J_3 = x\frac{\mathrm{d}}{\mathrm{d}x} - j \qquad J_+ = x\left[2j - x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = x[j - J_3]_q$$
$$J_- = \frac{1}{x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right]_q = \frac{1}{x}[j + J_3]_q. \tag{2.2}$$

The canonical basis of the space $\mathcal{H}_j$ consists of monomials

$$f_m^j(x) = c_m^j x^{j+m} \qquad c_m^j = q^{(m^2-j^2)/4}\begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} \tag{2.3}$$

where the q-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is defined as (we employ the standard notations of q-analysis; see, for example, [9, 10])

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_n(q;q)_{m-n}} = (-1)^n q^{mn-n(n-1)/2}\frac{(q^{-m};q)_n}{(q;q)_n} \tag{2.4}$$

and

$$(z;q)_0 = 1 \qquad (z;q)_n = \prod_{s=0}^{n-1}(1 - zq^s) \qquad n = 1, 2, 3, \ldots. \tag{2.5}$$

We equip the space $\mathcal{H}_j$ with the scalar product for which the monomials $f_m^j(x)$, $m = -j, -j+1, \ldots, j$, form an orthonormal basis of $\mathcal{H}_j$.

In the canonical basis (2.3) the operators J_3 and $J_\pm$ act as

$$J_3 f_m^j(x) = m f_m^j(x) \qquad J_\pm f_m^j(x) = [j \pm m + 1]_q^{1/2}[j \mp m]_q^{1/2} f_{m\pm 1}^j(x). \tag{2.6}$$

Obviously, the operator J_3 is diagonal in the canonical basis. We are interested in diagonalizing other operators from the representations T_j.

It was shown in [1, 2] that one can diagonalize the operator

$$\widetilde{J}_1 := \tfrac{1}{2}q^{J_3/4}(J_+ + J_-)q^{J_3/4} = \tfrac{1}{2}\big(q^{1/4}J_+ + q^{-1/4}J_-\big)q^{J_3/2}.$$

In addition to the operator $\widetilde{J}_1$, it is in fact natural to consider another operator $\widetilde{J}_2$, defined as

$$\widetilde{J}_2 = -\mathrm{i}[J_3, \widetilde{J}_1] = \frac{1}{2\mathrm{i}}q^{J_3/4}(J_+ - J_-)q^{J_3/4}.$$

The operators $\widetilde{J}_1$, $\widetilde{J}_2$ and J_3 obey the commutation relations

$$[J_3, \widetilde{J}_1] = \mathrm{i}\widetilde{J}_2 \qquad [\widetilde{J}_2, J_3] = \mathrm{i}\widetilde{J}_1 \qquad [\widetilde{J}_1, \widetilde{J}_2] = \mathrm{i}\widetilde{J}_3$$

where

$$\widetilde{J}_3 := \tfrac{1}{2}q^{J_3/2}\big(q^{1/2}J_-J_+ - q^{-1/2}J_+J_-\big)q^{J_3/2}.$$

Note that the operator $\widetilde{J}_3$ is diagonal in the canonical basis (2.3), namely,

$$\widetilde{J}_3 f_m^j(x) = \tfrac{1}{2}q^m\big(\{2j+1\}_q - q^m\{1\}_q\big)f_m^j(x)$$

where

$$\{A\}_q := \frac{q^{A/2} + q^{-A/2}}{q^{1/2} - q^{-1/2}}.$$

Our aim in the next section is to diagonalize the operators

$$\check{J}_1 := \widetilde{J}_1 + cq^{J_3} \equiv \tfrac{1}{2}\big(q^{1/4}J_+ + q^{-1/4}J_-\big)q^{J_3/2} + cq^{J_3} \tag{2.7}$$

$$\check{J}_2 := \widetilde{J}_2 + cq^{J_3} \equiv \frac{1}{2\mathrm{i}}\big(q^{1/4}J_+ - q^{-1/4}J_-\big)q^{J_3/2} + cq^{J_3}. \tag{2.8}$$

3. Nonstandard bases and diagonalization of the operators

It proves convenient to represent the real parameter c in (2.7) and (2.8) in the form

$$c = \frac{1}{2}[2\sigma]_q \equiv \frac{1}{2}\frac{q^{\sigma} - q^{-\sigma}}{q^{1/2} - q^{-1/2}}$$

and to denote the operators $\breve{J}_1$ and $\breve{J}_2$ as $\breve{J}_1^{(\sigma)}$ and $\breve{J}_2^{(\sigma)}$, respectively, in order to indicate their dependence on σ. We begin with eigenfunctions and eigenvalues of the operator $\breve{J}_1^{(\sigma)}$, defined by (2.7).

By analogy with the operator $\widetilde{J}_1$ (see [1]), let us look for eigenfunctions of $\breve{J}_1^{(\sigma)}$ of the form

$$\chi_m^j(x;\sigma) = (\alpha x; q)_{j-m}(-\beta x; q)_{j+m} \tag{3.1}$$

where α and β do not depend on x, but may be q and σ dependent. It is assumed in (3.1) that m is an integer or a half-integer such that $-j \leqslant m \leqslant j$ and $j - m \in \mathbb{Z}$. We need to find the explicit form of α and β from the requirement that

$$\breve{J}^{(\sigma)}\chi_m^j(x;\sigma) = \tfrac{1}{2}[2M]_q\chi_m^j(x;\sigma)$$

where $[2M]_q/2$ are some eigenvalues. Since

$$q^{cx\frac{\mathrm{d}}{\mathrm{d}x}} f(x) = f(q^c x) \tag{3.2}$$

for any real c and an arbitrary function $f(x)$, we have

$$q^{J_3}\chi_m^j(x;\sigma) = q^{-j}\chi_m^j(qx;\sigma). \tag{3.3}$$

From (2.2) and (3.2) it also follows that

$$J_+ q^{J_3/2}\chi_m^j(x;\sigma) = \frac{q^{-j/2}x}{q^{1/2} - q^{-1/2}}\left\{q^j\chi_m^j(x;\sigma) - q^{-j}\chi_m^j(qx;\sigma)\right\}. \tag{3.4}$$

Similarly,

$$J_- q^{J_3/2}\chi_m^j(x;\sigma) = \frac{q^{-j/2}}{x\left(q^{1/2} - q^{-1/2}\right)}\left\{\chi_m^j(qx;\sigma) - \chi_m^j(x;\sigma)\right\}. \tag{3.5}$$

From the definition (2.5) of the q-shifted factorial $(z;q)_n$ it is easy to deduce that

$$(qz;q)_n = \frac{1 - zq^n}{1 - z}(z;q)_n. \tag{3.6}$$

Therefore

$$\chi_m^j(qx;\sigma) = \frac{(1 - \alpha x q^{j-m})(1 + \beta x q^{j+m})}{(1 - \alpha x)(1 + \beta x)}\chi_m^j(x;\sigma). \tag{3.7}$$

In accordance with (2.7), it now remains only to multiply (3.3) by $[2\sigma]_q/2$, (3.4) by $q^{1/4}/2$ and (3.5) by $q^{-1/4}/2$, and then sum up the results. Then we obtain the following equation:

$$\breve{J}_1^{(\sigma)}\chi_m^j(x;\sigma) = \frac{Ax^2 + Bx + C}{2\left(q^{1/2} - q^{-1/2}\right)(1 - \alpha x)(1 + \beta x)}\chi_m^j(x;\sigma) \tag{3.8}$$

where the constant coefficients A, B and C are equal to

$$\begin{aligned} A &= q^{(2j+1)/4}(\beta - \alpha) - q^j(q^{\sigma} - q^{-\sigma})\alpha\beta - (\beta q^m - \alpha q^{-m})q^{(1-2j)/4} \\ B &= q^{(2j+1)/4}(1 - q^{-2j})\left(1 - q^{j-1/2}\alpha\beta\right) + (q^{\sigma} - q^{-\sigma})(\beta q^m - \alpha q^{-m}) \\ C &= q^{-(2j+1)/4}(\alpha - \beta) + q^{-j}(q^{\sigma} - q^{-\sigma}) + q^{(2j-1)/4}(\beta q^m - \alpha q^{-m}) \end{aligned} \tag{3.9}$$

respectively. We will obtain eigenvalues of the operator $\breve{J}_1^{(\sigma)}$ on the right-hand side of (3.8), provided that $Ax^2+Bx+C = C(1-\alpha x)(1+\beta x)$. This relation is equivalent to two equations: $A = -\alpha\beta C$ and $B = (\beta-\alpha)C$. For each fixed integer or half-integer m such that $-j \leqslant m \leqslant j$ and $j-m \in \mathbb{Z}$, the only solution of this inhomogeneous system (see (3.9)) of equations in α and β is $\alpha = q^{(1-2j)/4-\sigma}$ and $\beta = q^{(1-2j)/4+\sigma}$. Consequently, $C = q^{m+\sigma} - q^{-m-\sigma}$. We may thus formulate the following assertion.

For any real σ the operator $\breve{J}_1^{(\sigma)}$ can be diagonalized and its eigenfunctions are

$$\chi_m^j(x;\sigma) = \left(q^{(1-2j)/4-\sigma}x;q\right)_{j-m}\left(-q^{(1-2j)/4+\sigma}x;q\right)_{j+m}. \tag{3.10}$$

The functions $\chi_m^j(x;\sigma)$, $m = -j, -j+1, \ldots, j$, are linearly independent because they correspond to distinct eigenvalues $[2(m+\sigma)]_q/2$:

$$\breve{J}_1^{(\sigma)}\chi_m^j(x;\sigma) = \tfrac{1}{2}[2(m+\sigma)]_q\chi_m^j(x;\sigma). \tag{3.11}$$

The number of these eigenfunctions coincides with a dimension of the representation T_j. Therefore, these functions constitute a nonstandard basis of the representation space $\mathcal{H}_j$.

Let us understand now how the eigenfunctions $\chi_m^j(x;\sigma)$ are related to the canonical basis $f_m^j(x) = c_m^j x^{j+m}$. This can be easily derived using a generating function for the dual q-Krawtchouk polynomials. We remind the reader that the terminating basic hypergeometric series ${}_{k+1}\phi_k$ with $k+1$ numerator parameters $q^{-n}, a_1, \ldots, a_k$ and k denominator parameters $b_1, \ldots, b_k$ is explicitly given by the formula (see [9] or [10])

$${}_{k+1}\phi_k\left(\left.\begin{matrix} q^{-n}, a_1, \ldots, a_k \\ b_1, \ldots, b_k\end{matrix}\right| q; z\right) = \sum_{j=0}^{n}\frac{(q^{-n};q)_j(a_1;q)_j\cdots(a_k;q)_j}{(q;q)_j(b_1;q)_j\cdots(b_k;q)_j}z^j. \tag{3.12}$$

Therefore the relation (see [10], p 101)

$$\begin{aligned} K_n(\lambda(t);c,N|q) &:= {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, q^{-t}, & cq^{t-N} \\ q^{-N}, & 0\end{matrix}\right| q; q\right) \\ &= \frac{(q^{t-N};q)_n q^{-nt}}{(q^{-N};q)_n}{}_2\phi_1\left(\left.\begin{matrix} q^{-n}, q^{-t} \\ q^{N-t-n+1}\end{matrix}\right| q; cq^{t+1}\right) \end{aligned} \tag{3.13}$$

where $\lambda(t) = q^{-t} + cq^{t-N}$, for $n = 0, 1, \ldots, N$, defines the dual q-Krawtchouk polynomials, which are orthogonal on the set $t \in \{0, 1, \ldots, N\}$. They have a generating function of the form (see [10], p 103)

$$(q^{-N}t;q)_{N-k}(cq^{-N}t;q)_k = \sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}K_n(\lambda(k);c,N|q)t^n \qquad 0 \leqslant k \leqslant N. \tag{3.14}$$

From (3.14) it is obvious that the eigenfunctions (3.10) can be written as

$$\chi_m^j(x;\sigma) = \sum_{n=0}^{2j} q^{n(n+2\sigma-j-1/2)/2}\begin{bmatrix} 2j \\ n\end{bmatrix}_q K_n(\lambda(j-m); -q^{-2\sigma}, 2j|q)x^n \tag{3.15}$$

using relation (2.4) for the q-binomial coefficient. Combining (2.3) with (3.15), we get

$$\chi_m^j(x;\sigma) = \sum_{m'=-j}^{j} q^{(j+m')(j+m'+4\sigma-1)/4}\begin{bmatrix} 2j \\ j+m'\end{bmatrix}_q^{1/2} K_{j+m'}(\lambda(j-m); -q^{-2\sigma}, 2j|q)f_{m'}^j(x). \tag{3.16}$$

The basis $\left\{\chi_m^j(x;\sigma)\right\}_{-j}^{j}$ is orthogonal but not normalized. The functions

$$\hat{\chi}_m^j(x;\sigma) := c_m\chi_m^j(x;\sigma) \qquad m = -j, -j+1, \ldots, j$$

where

$$c_m = (-q^{-\sigma})^{j-m} q^{(j-m)(3j+m)/2} \times \left(\frac{(-q^{-2\sigma-2j}; q)_{j-m} (q^{-2j}; q)_{j-m} (1 + q^{-2\sigma-2m})}{(-q^{2\sigma}; q)_{2j} (q; q)_{j-m} (-q^{2\sigma+1}; q)_{j-m} (1 + q^{-2\sigma-2j})} \right)^{1/2} \tag{3.17}$$

form an orthonormal basis of $\mathcal{H}_j$. This is because the matrix $\left(a^j_{m'm}\right)$ with entries

$$a^j_{m'm} = c_m q^{(j+m')(j+m'+4\sigma-1)/4} \begin{bmatrix} 2j \\ j+m' \end{bmatrix}_q^{1/2} K_{j+m'}(\lambda(j-m); -q^{-2\sigma}, 2j|q) \tag{3.18}$$

which connects the bases $\left\{\hat{\chi}^j_m(x;\sigma)\right\}^j_{-j}$ and $\left\{f^j_{m'}\right\}^j_{-j}$, is unitary (due to the orthogonality relation (3.17.2) in [10] for the dual q-Krawtchouk polynomials).

Remark. Note that we could derive (by using the connection (3.16) between the bases $\left\{f^j_{m'}\right\}$ and $\left\{\chi^j_m\right\}$) the orthogonality relation for the dual q-Krawtchouk polynomials. This derivation could be based on the use of the theory of selfadjoint operators, which can be represented by Jacobi matrices (see chapter VII in [11], in particular, formula (1.21) in this chapter). However, we have not engaged into such derivation here for the following reason. The dual q-Krawtchouk polynomials are a special case of the more general q-Racah polynomials and there exists a simple proof of the orthogonality relation for the latter (and, consequently, for the former) polynomials by means of the Racah coefficients for the algebra $su_q(2)$ (see [5], vol 3, section 14.6.5, formula (5)).

Eigenfunctions and eigenvalues of the operator $\breve{J}^{(\sigma)}_2$ can be obtained in exactly the same way as for the operator $\breve{J}^{(\sigma)}_1$. Therefore, we only state the result. Eigenfunctions of $\breve{J}^{(\sigma)}_2$ have the form

$$\xi^j_m(x;\sigma) = (\mathrm{i}\alpha x; q)_{j-m}(-\mathrm{i}\beta x; q)_{j+m} = \chi^j_m(\mathrm{i}x;\sigma) \tag{3.19}$$

where $\alpha = q^{(1-2j)/4-\sigma}$ and $\beta = q^{(1-2j)/4+\sigma}$. The eigenvalues of $\breve{J}^{(\sigma)}_2$ coincide with those of the operator $\breve{J}^{(\sigma)}_1$:

$$\breve{J}^{(\sigma)}_2 \xi^j_m(x;\sigma) = \tfrac{1}{2}[2(m+\sigma)]_q \xi^j_m(x;\sigma).$$

The functions $\hat{\xi}^j_m(x;\sigma) := c_m \xi^j_m(x;\sigma)$, where $m = -j, -j+1, \ldots, j$ and the c_m are the same as in (3.17), form an orthonormal basis of $\mathcal{H}_j$. The eigenfunctions $\hat{\xi}^j_m(x;\sigma)$ are connected with the canonical basis by the formula $\hat{\xi}^j_m(x;\sigma) = \sum_{m'=-j}^j b^j_{m'm} f^j_{m'}(x)$, where $b^j_{m'm} = (-\mathrm{i})^{j'+m} a^j_{m'm}$ and $a^j_{m'm}$ are given by (3.18).

4. Intertwining operator

In this section we derive an explicit form of an operator $B^j(\sigma)$ that intertwines $\breve{J}^{(\sigma)}_1$ and $[2(J_3+\sigma)]_q/2$. Using definition (2.7) of the operator $\breve{J}^{(\sigma)}_1$ and formulae (2.6) for the action of the operators J_3 and $J_\pm$ on the canonical basis, we write the matrix form of the relation

$$2\breve{J}^{(\sigma)}_1 B^j(\sigma) = B^j(\sigma)[2(J_3+\sigma)]_q$$

in the canonical basis (2.3). As a result we obtain an equation which relates the matrix elements $B^j_{mm'}(\sigma)$ of the operator $B^j(\sigma)$ in the canonical basis:

$$q^{(2m-1)/4}\sqrt{[j+m]_q[j-m+1]_q}\, B^j_{m-1,m'}(\sigma) + q^{(2m+1)/4}\sqrt{[j-m]_q[j+m+1]_q}\, B^j_{m+1,m'}(\sigma) = ([2(m'+\sigma)]_q - q^m[2\sigma]_q) B^j_{m,m'}(\sigma). \tag{4.1}$$

To dispense with square roots in (4.1), we make the substitution

$$B^j_{m,m'}(\sigma) = q^{m\sigma+[m^2-j^2+m(2j-1)]/4} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} \widetilde{B}^j_{m,m'}(\sigma). \tag{4.2}$$

After some simplification, we obtain for $\widetilde{B}^j_{m,m'}(\sigma)$ the recurrence relation

$$\begin{aligned}(1-q^{m-j})\widetilde{B}^j_{m+1,m'}(\sigma) - q^{-2j-2\sigma}(1-q^{j+m})\widetilde{B}^j_{m-1,m'}(\sigma) \\ = q^{-j}[q^{m'} - q^m + (q^m - q^{-m'})q^{-2\sigma}]\widetilde{B}^j_{m,m'}(\sigma).\end{aligned} \tag{4.3}$$

If one compares it with the recurrence relation (see [10], p 102)

$$\begin{aligned}(1-q^{n-N})K_{n+1}(\lambda(t); c, N|q) + cq^{-N}(1-q^n)K_{n-1}(\lambda(t); c, N|q) \\ = [\lambda(t) - (1+c)q^{n-N}]K_n(\lambda(t); c, N|q)\end{aligned}$$

for the dual q-Krawtchouk polynomials with the parameters

$$\begin{array}{lll} n = j+m & N = 2j & c = -q^{-2\sigma} \\ t = j-m' & \lambda(j-m') = q^{m'-j} - q^{-2\sigma-j-m'} & \end{array}$$

then it becomes evident that

$$\widetilde{B}^j_{m,m'}(\sigma) = K_{j+m}(\lambda(j-m'); -q^{-2\sigma}, 2j|q)b_j(m') \tag{4.4}$$

where $b_j(m')$ is some arbitrary function of the index m'. Substituting (4.4) into (4.2), we obtain

$$B^j_{m,m'}(\sigma) = q^{m\sigma+[m^2-j^2+m(2j-1)]/4} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} K_{j+m}(\lambda(j-m'); -q^{-2\sigma}, 2j|q)b_j(m'). \tag{4.5}$$

Since intertwining operators are always defined up to the multiplication by an arbitrary diagonal operator and $K_0(x(s); c, N|q) = 1$ by the initial condition, it is convenient to represent (4.5) in the form (cf (3.15))

$$B^j_{m,m'}(\sigma) = q^{(j+m)[\sigma+(j+m-1)/4]} \begin{bmatrix} 2j \\ j+m \end{bmatrix}_q^{1/2} K_{j+m}(\lambda(j-m'); -q^{-2\sigma}, 2j|q)B^j_{-j,m'}(\sigma). \tag{4.6}$$

An intertwining operator for the operators $\breve{J}_2^{(\sigma)}$ and $[2(J_3+\sigma)]_q/2$ is constructed in exactly the same way.

5. Representation operators in nonstandard bases

We have found new (nonstandard) bases in the space of the representation T_j and have determined how the operator $\breve{J}_1^{(\sigma)}$ acts upon the corresponding basis. Let us now calculate how the operator q^{-J_3} acts upon this basis.

Using formulae (3.3) and (3.13), we find that

$$\begin{aligned}q^{-J_3}\chi^j_m(x;\sigma) &= q^j\chi^j_m(q^{-1}x;\sigma) \\ &= q^j \sum_{n=0}^{2j} q^{n(n+2\sigma-j-1/2)/2} \begin{bmatrix} 2j \\ n \end{bmatrix}_q q^{-n} K_n(\lambda(j-m); -q^{-2\sigma}, 2j|q)x^n.\end{aligned} \tag{5.1}$$

The dual q-Krawtchouk polynomials in this expression satisfy the q-difference equation

$$q^{-n}K_n(\lambda(t)) = B(t)K_n(\lambda(t+1)) + [1-B(t)-D(t)]K_n(\lambda(t)) + D(t)K_n(\lambda(t-1)) \tag{5.2}$$

where $K_n(\lambda(t)) \equiv K_n(\lambda(t); c, N|q)$ and

$$B(t) = \frac{(1-q^{t-N})(1-cq^{t-N})}{(1-cq^{2t-N})(1-cq^{2t-N+1})} \qquad D(t) = \frac{cq^{2t-2n-1}(1-q^t)(1-cq^t)}{(1-cq^{2t-N-1})(1-cq^{2t-N})} \tag{5.3}$$

(see formula (3.17.5) in [10]). Substituting the right-hand side of (5.2) into (5.1), we obtain the following formula for the action of the operator q^{-J_3}:

$$\begin{aligned} q^{-J_3}\chi^j_m(x;\sigma) &= q^j B_\sigma(j,m)\chi^j_{m-1}(x;\sigma) + q^j D_\sigma(j,m)\chi^j_{m+1}(x;\sigma) \\ &\quad + q^j[1 - B_\sigma(j,m) - D_\sigma(j,m)]\chi^j_m(x;\sigma) \end{aligned} \tag{5.4}$$

where, in accordance with (5.3),

$$\begin{aligned} B_\sigma(j,m) &= -q^{2m+2\sigma-2j-1}\frac{(1-q^{j+m})(1+q^{j+m+2\sigma})}{(1+q^{2m+2\sigma})(1+q^{2m+2\sigma-1})} \\ D_\sigma(j,m) &= -q^{2m+2\sigma-2j}\frac{(1-q^{j-m})(1+q^{j-m-2\sigma})}{(1+q^{2m+2\sigma})(1+q^{2m+2\sigma+1})}. \end{aligned}$$

Direct calculation shows that the expression for $1 - B_\sigma(j,m) - D_\sigma(j,m)$ in (5.4) can be represented as

$$\begin{aligned} &1 - B_\sigma(j,m) - D_\sigma(j,m) \\ &\quad = q^{m-j}\frac{(1-q^{2\sigma})(1-q^{2\sigma+2m}) + q^{2\sigma+m}(q^{j+1/2}+q^{-j-1/2})(q^{1/2}+q^{-1/2})}{(1+q^{2m+2\sigma+1})(1+q^{2m+2\sigma-1})}. \end{aligned}$$

Thus, we have the action formulae for the operators $\breve{J}^{(\sigma)}_1$ and q^{-J_3} in the nonstandard basis. By using these operators we may find an explicit form of many other representation operators with respect to the nonstandard basis. For example, we have an explicit form of the operators

$$\begin{aligned} \breve{J}^{(\sigma)}_1 q^{-J_3} - \tfrac{1}{2}[2\sigma]_q &= \left(q^{1/4}J_+ + q^{-1/4}J_-\right)q^{-J_3/2} \\ q^{-J_3}\breve{J}^{(\sigma)}_1 - \tfrac{1}{2}[2\sigma]_q &= \left(q^{-3/4}J_+ + q^{3/4}J_-\right)q^{-J_3/2}. \end{aligned}$$

Therefore, we know how the operator

$$q^{-J_3}\breve{J}^{(\sigma)}_1 - q\breve{J}^{(\sigma)}_1 q^{-J_3} - \tfrac{1}{2}[2\sigma]_q(1-q) = q^{1/4}(q^{-1}-q)J_+q^{-J_3/2}$$

acts. Similarly, we obtain the operators $J_-q^{-J_3/2}$, $q^{-J_3/2}J_+$ and $q^{-J_3/2}J_-$. However, we do not know yet how to calculate arbitrary representation operators by means of simple algebraic operations with $\breve{J}^{(\sigma)}_1$ and q^{-J_3}. In order to calculate any representation operator, we have to find the action of the inverse operator q^{J_3} . Unfortunately, we could not find an explicit form for the action of this operator in an irreducible representation of $su_q(2)$.

6. Diagonalization of other representation operators

We can also diagonalize other operators of the representation T_j, which are related to families of q-orthogonal polynomials from higher levels in the Askey-scheme [10]. We consider in this section the case of a one-parameter family of operators, which are diagonalized by means of the dual q-Hahn polynomials. Let us introduce an operator (cf (2.7) and (2.8))

$$I^{(\sigma)} := (\sigma J_+ - \sigma^{-1}J_-)q^{J_3/2} + \left(q^{1/4} - q^{-1/4}\right)^{-1}q^{J_3} \tag{6.1}$$

where $\sigma \in \mathbb{C}$. We shall be looking for eigenfunctions of $I^{(\sigma)}$ of the form

$$\eta^j_m(x;\sigma) = \left(ax; q^{1/2}\right)_{j-m}(bx;q)_{j+m}. \tag{6.2}$$

The action of the operators q^{J_3}, $J_+q^{J_3/2}$ and $J_-q^{J_3/2}$ on the functions $\eta_m^j(x;\sigma)$ is given by formulae (3.3)–(3.5), respectively. Therefore, it is immediate that

$$I^{(\sigma)}\eta_m^j(x;\sigma) = \frac{q^{-j/2}}{\sigma x\left(q^{1/2}-q^{-1/2}\right)}\left\{(1+\sigma^2 q^j x^2)\eta_m^j(x,\sigma)\right.$$
$$\left. -\left[1-\sigma\left(q^{1/4}+q^{-1/4}\right)q^{-j/2}x+\sigma^2 q^{-j}x^2\right]\eta_m^j(qx;\sigma)\right\}. \tag{6.3}$$

Taking into account relation (3.6), one can express the function $\eta_m^j(qx;\sigma)$ in terms of the $\eta_m^j(x;\sigma)$:

$$\eta_m^j(qx;\sigma) = \frac{\left(1-axq^{(j-m+1)/2}\right)\left(1-axq^{(j-m)/2}\right)\left(1-bxq^{j+m}\right)}{\left(1-axq^{1/2}\right)(1-ax)(1-bx)}\eta_m^j(x;\sigma). \tag{6.4}$$

Substituting (6.4) into (6.3), we obtain

$$I^{(\sigma)}\eta_m^j(x;\sigma) = \frac{q^{-j/2}}{\sigma\left(q^{1/2}-q^{-1/2}\right)}\frac{Ax^3+Bx^2+Cx+D}{\left(1-axq^{1/2}\right)(1-ax)(1-bx)}\eta_m^j(x;\sigma) \tag{6.5}$$

where the coefficients A, B, C and D are equal to

$$\begin{aligned}
A &= \sigma^2 a^2 q^{1/2}(q^j-q^{-m}) + \sigma ab\left(1+q^{1/2}\right)q^{j/2}\left[\sigma\left(q^{j/2}-q^{m/2}\right)-aq^{j+1/4}\right]\\
B &= \sigma a\left(1+q^{1/2}\right)\left[aq^{-m+(2j+1)/4}+\sigma\left(q^{-(j+m)/2}-q^j\right)+bq^{j+m/2}\left(q^{1/4}+q^{-1/4}\right)\right]\\
&\quad +\sigma^2 b(q^m-q^j)+a^2bq^{1/2}(q^{2j}-1)\\
C &= \sigma^2(q^j-q^{-j})+a^2q^{1/2}(1-q^{j-m})+\left(1+q^{1/2}\right)\left[ab\left(1-q^{(3j+m)/2}\right)\right.\\
&\quad \left. -\sigma q^{-1/4}\left(bq^{m+j/2}+a\left(1+q^{1/2}\right)q^{-m/2}\right)\right]\\
D &= \sigma q^{-j/2}\left(q^{1/4}+q^{-1/4}\right)+a\left(1+q^{1/2}\right)\left(q^{(j-m)/2}-1\right)+b(q^{j+m}-1)
\end{aligned} \tag{6.6}$$

respectively. From (6.5) it is obvious that the relation

$$Ax^3+Bx^2+Cx+D = D\left(1-axq^{1/2}\right)(1-ax)(1-bx) \tag{6.7}$$

must hold in order to obtain eigenvalues on the right-hand side of (6.5). The requirement (6.7) is equivalent to equations

$$A = -a^2bq^{1/2}D \qquad B = a\left[b+q^{1/2}(a+b)\right]D \qquad C = -\left[\left(1+q^{1/2}\right)a+b\right]D \tag{6.8}$$

which have the unique solution

$$a = \sigma q^{-(2j+1)/4} \qquad b = \sigma q^{(1-2m)/4}. \tag{6.9}$$

Consequently,

$$D = \sigma q^{j/2}\left(q^{(2j+2m+1)/4}+q^{-(2j+2m+1)/4}\right).$$

We may formulate thus the following:

For any value of σ the operator $I^{(\sigma)}$ can be diagonalized and its eigenfunctions are

$$\eta_m^j(x;\sigma) = \left(\sigma q^{-(2j+1)/4}x;q^{1/2}\right)_{j-m}\left(\sigma q^{(1-2m)/4}x;q\right)_{j+m}. \tag{6.10}$$

The functions $\eta_m^j(x;\sigma), m=-j,-j+1,\ldots,j$, are linearly independent because they correspond to distinct eigenvalues $\{j+m+1/2\}_q$:

$$I^{(\sigma)}\eta_m^j(x;\sigma) = \{j+m+1/2\}_q\eta_m^j(x;\sigma). \tag{6.11}$$

The number of these eigenfunctions coincides with a dimension of the representation T_j. For this reason, these functions constitute another nonstandard basis of the representation space $\mathcal{H}_j$.

Let us now discover how the eigenfunctions $\eta^j_m(x;\sigma)$ are related to the canonical basis $f^j_m(x)$. This time we will need a generating function for the dual q-Hahn polynomials (see [10], p 78):

$$R_n(\lambda(x);\gamma,\delta,N|q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-x}, \gamma\delta q^{x+1} \\ \gamma q, q^{-N}\end{matrix}\middle| q;q\right) \tag{6.12}$$

where $\lambda(x) := q^{-x} + \gamma\delta q^{x+1}$ and $n = 0, 1, 2, \ldots, N$. These polynomials have a generating function of the form (see [10], p 79)

$$(q^{-N}t;q)_{N-k}\cdot {}_2\phi_1\left(\begin{matrix} q^{-k}, \delta^{-1}q^{-k} \\ \gamma q\end{matrix}\middle| q;\gamma\delta q^{k+1}t\right) = \sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}R_n(\lambda(k);\gamma,\delta,N|q)t^n \qquad k = 0,1,2\ldots,N. \tag{6.13}$$

By using the well-known relations $(a;q)_n(-a;q)_n = (a^2;q^2)_n$ and ${}_1\phi_0(q^{-n};q,z) = (zq^{-n};q)_n$, one can show that in the particular case of $\gamma = \delta = -1$ the generating function (6.13) reduces to the expression

$$(q^{-N}t;q)_{N-k}(q^{1-k}t;q^2)_k = \sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q,q)_n}R_n(\lambda(k);-1,-1,N|q)t^n. \tag{6.14}$$

From (6.14) it is now immediate that

$$\eta^j_m(x;\sigma) = \sum_n \frac{(q^{-j};q^{1/2})_n}{(q^{1/2};q^{1/2})_n}R_n\big(\lambda(j+m);-1,-1,2j|q^{1/2}\big)\big(\sigma q^{(2j-1)/4}x\big)^n. \tag{6.15}$$

Taking into account the explicit form of the canonical basis (2.3), relation (6.15) is equivalent to the expansion

$$\eta^j_m(x;\sigma) = \sum_{m'}\alpha^j_{m'm}(\sigma;q)f^j_{m'}(x) \tag{6.16}$$

where the connection coefficients $\alpha^j_{m'm}(\sigma;q)$ are equal to

$$\alpha^j_{m'm}(\sigma;q) = \big(-q^{-1/2}\sigma\big)^{j+m'}\frac{\left[\begin{smallmatrix}2j\\ j+m'\end{smallmatrix}\right]_{q^{1/2}}}{\left[\begin{smallmatrix}2j\\ j+m'\end{smallmatrix}\right]_q^{1/2}}R_{j+m'}\big(\lambda(j+m);-1,-1,2j|q^{1/2}\big). \tag{6.17}$$

The basis $\{\eta^j_m(x;\sigma)\}^j_{-j}$ is orthogonal but not normalized. At $\sigma = -\mathrm{i}$ these basis elements can be normalized with the aid of the orthogonality relation for q-Hahn polynomials (see formula (7.2.22) in [9]). In this case, the functions

$$\hat{\eta}^j_m(x;-\mathrm{i}) = c_m\eta^j_m(x;-\mathrm{i})$$

with

$$c_m = \left(\frac{(-q^{1/2};q^{1/2})_{2j}(q^{-j};q^{1/2})_n(1-q^{n+1/2})}{(q;q^{1/2})_{2j}(q^{j+1};q^{1/2})_n(1-q)q^{n/2}}\right)^{1/2} q^{n[j-(n-1)/4]} \qquad n = j+m$$

form an orthonormal basis of the space $\mathcal{H}_j$. The matrix $\big(\hat{a}^j_{m',m}(\sigma,-\mathrm{i})\big)^j_{m',m=-j}$ with $\hat{a}^j_{m',m}(\sigma,-\mathrm{i}) = c_m a^j_{m',m}(\sigma,-\mathrm{i})$, which connects the bases $\{f^j_{m'}\}$ and $\{\hat{\eta}^j_m\}$, is unitary.

We can also evaluate the action of the operator $q^{-J_3/2}$ on the basis functions $\eta^j_m(x;\sigma)$. To this end we use the q-difference equation (3.7.5) from [10] for the dual q-Hahn polynomials. For $R_n(\lambda(t)) \equiv R_n\big(\lambda(t);-1,-1,2j|q^{1/2}\big)$, $t = j+m$, this equation can be written as

$$q^{-n/2}R_n(\lambda(t)) = \frac{1-q^{(t-2j)/2}}{1-q^{(2t+1)/2}}R_n(\lambda(t+1)) + \frac{q^{(t-2j)/2}-q^{(2t+1)/2}}{1-q^{(2t+1)/2}}R_n(\lambda(t-1)). \tag{6.18}$$

Since $q^{-J_3/2}x^n = q^{-(n-j)/2}x^n$, one can apply equation (6.18) to expansion (6.15) and obtain the formula

$$q^{-J_3/2}\eta_m^j(x;\sigma) = q^{j/2}\frac{1-q^{(m-j)/2}}{1-q^{j+m+1/2}}\eta_{m+1}^j(x;\sigma) + q^{j/2}\frac{q^{(j+m)/2}\left(q^{-j}-q^{(m+j+1)/2}\right)}{1-q^{j+m+1/2}}\eta_{m-1}^j(x;\sigma).$$

By using the operators $I^{(\sigma)}$ and $q^{-J_3/2}$, one can find explicit forms of some other operators of the representation T_j (cf section 5).

7. Concluding remarks

In sections 3 and 6 we have explicitly constructed nonstandard bases (3.10) and (6.10), respectively, for representations of the quantum algebra $su_q(2)$. The relations between the standard and nonstandard basis functions are, respectively, given by equations (3.16) and (6.16).

The representations were realized in a space of functions of one variable. As was already emphasized in [1], it would be of interest to consider other realizations, for instance, in spaces of functions on a two-dimensional sphere, as in [12].

From the theory of representations of Lie groups it is known that for some cases connection coefficients between two bases for a fixed irreducible representation of some Lie group G are at the same time Clebsch–Gordan coefficients for a tensor product of irreducible representations of another Lie group G' (see, for example, [5], vol 2, section 12.3.5). There exist similar phenomena in the case of quantum groups as well. So one of the referees of our paper raised the question whether the coefficients (3.18) are also Clebsch–Gordan coefficients of some other algebra. The point is that the coefficients (3.18) diagonalize the operator (2.7), which contains the term cq^{J_3}. This circumstance complicates the problem of finding such a connection. Therefore this problem will be studied separately.

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